Lagrangian Gauge Structure Functions for Systems with First-Class Constraints

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Abstract

The structure functions of the lagrangian gauge algebra are given explicitly in terms of the hamiltonian constraints and the first-order hamiltonian structure functions and their derivatives.

The importance of gauge symmetries in all modern relativistic theories of the fundamental interactions is well understood at present. Electromagnetic and Yang-Mills theories are examples of systems with natural bases of gauge generators that form closed lagrangian gauge algebras. The hamiltonian first-class constraints in these theories are linear in the canonical momenta. Their first-order hamiltonian gauge structure functions are constants and therefore the first-class constraints form Lie groups. In Einstein's theory of gravity one of the hamiltonian constraints is quadratic in the momenta [1] and the first-order hamiltonian gauge structure functions do not depend on the canonical momenta but do depend on the metric [2]. The hamiltonian constraints in Einstein's theory do not form a Lie group [2], but the natural basis of lagrangian gauge generators does form a closed gauge algebra.

The study of open lagrangian gauge algebras started with the discovery of supergravity [3, 4, 5]. In theories with open lagrangian gauge algebras we have to deal with lagrangian structure functions of higher orders. The existence of these structure functions has been established using an axiomatic approach in [6]. In the quantum theory, in order to construct the Feynman diagrams one needs first to determine all the gauge structure functions. Although these functions have been found explicitly in particular cases [4], their general form for the generic case of a hamiltonian system with first-class constraints is not known. The purpose of this paper is to solve this problem. We will present explicit expressions for the lagrangian gauge structure functions up to fourth order. We will show how these structure tensors are determined by the hamiltonian constraints and the hamiltonian first-order structure functions. It is remarkable that to determine the higher order lagrangian gauge structure tensors no knowledge of the higher-order hamiltonian structure functions is required. They depend only on the zeroth- and first-order hamiltonian structure functions. The method presented here can be used to find the structure functions of even higher levels.

Let us consider a physical system described by the action:

$$S = \int dt L \tag{1}$$

L is the Lagrangian of the system, which is a function defined in the velocity phase space TQ (TQ is the tangent bundle of the n-dimensional configuration space Q). The variables q^i $(i = 1, 2, ..., n)$ are the generalized coordinates and $\dot{q}^i$ the generalized velocities.

The Euler-Lagrange equations of motion may be written in the following form:

$$L_i \equiv W_{ij}\ddot{q}^j - \alpha_i = 0 \tag{2}$$

where,

$$W_{ij} \equiv \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \tag{3}$$

$$\alpha_i \equiv \frac{\partial L}{\partial q^i} - \dot{q}^l \frac{\partial^2 L}{\partial q^l \partial \dot{q}^i} \tag{4}$$

We assume that the dimension of the kernel of the Hessian matrix W is constant (the same in all points of TQ) and equal to m:

$$rank\|W\| = n - m \tag{5}$$

Let us denote as $R_\mu(q, \dot{q})$ ($\mu = 1, 2, ..., m$) a set of linearly independent null eigenvectors of the Hessian matrix:

$$R^i_\mu W_{ij} \equiv 0 \tag{6}$$

$$rank\|R^i_\mu\| = m \tag{7}$$

Using the Euler-Lagrange equations (2) and the identities (6) we obtain the lagrangian constraints of the first-level [7]:

$$\chi^{(1)}_\mu \equiv R^i_\mu \alpha_i = 0 \tag{8}$$

Let us assume that (8) do not bring about any restrictions in the velocity phase space TQ (we assume that (8) are identities in TQ):

$$R^i_\mu \alpha_i \equiv 0 \tag{9}$$

In this case, the left-hand-sides L_i of the Euler-Lagrange equations (2) satisfy the Noether identities [7]:

$$R^i_\mu(q, \dot{q}) L_i(q, \dot{q}, \ddot{q}) \equiv 0 \tag{10}$$

The Noether identities are satisfied by any trajectory $q(t)$ in the configuration space Q. According to the second Noether theorem [8], the action (1) is invariant under the infinitesimal gauge transformations:

$$\delta q^i(t) = \epsilon^\mu(t) R^i_\mu(q(t), \dot{q}(t)) \tag{11}$$

R^i_μ are the generators of the gauge transformations (11).

It is not difficult to prove that under infinitesimal gauge transformations of the form (11) the left-hand-sides of the Euler-Lagrange equations (2) transform as follows:

$$\delta L_i = \epsilon^\mu \left[-\frac{\partial R^j_\mu}{\partial q^i} L_j + \frac{d}{dt}\left(\frac{\partial R^j_\mu}{\partial \dot{q}^i} L_j\right) \right] + \dot{\epsilon}^\mu \frac{\partial R^j_\mu}{\partial \dot{q}^i} L_j \tag{12}$$

From (7) it follows that we are dealing with an irreducible gauge theory [5, 9]. The most general regular solution of the equation:

$$\lambda^i L_i \equiv 0 \tag{13}$$

can be written as [6]:

$$\lambda^i = R^i_\mu \epsilon^\mu + T^{ij} L_j \tag{14}$$

where,

$$T^{ij} = -T^{ji} \tag{15}$$

The trivial gauge transformations $T^{ij}L_j$ have no physical significance. However, in relativistic field theories trivial gauge transformations may appear as a result of requiring manifest covariance and preservation of locality.

Let us consider the commutator of two infinitesimal gauge transformations:

$$\delta_1\delta_2 q^i - \delta_2\delta_1 q^i \equiv [\delta_1, \delta_2]q^i \tag{16}$$

In general, the commutator of two infinitesimal gauge transformations of the form (11) is a gauge transformation of the form (14) [5, 9, 6]:

$$[\delta_1, \delta_2]q^i \equiv R^i_\gamma \epsilon^\gamma + E^{ij}L_j \tag{17}$$

where,

$$\epsilon^\gamma = T^\gamma_{\alpha\beta}(q, \dot{q})\epsilon^\alpha_1 \epsilon^\beta_2 \tag{18}$$

$$T^\gamma_{\alpha\beta} = -T^\gamma_{\beta\alpha} \tag{19}$$

$$E^{ij} = E^{ij}_{\alpha\beta}(q, \dot{q})\epsilon^\alpha_1 \epsilon^\beta_2 \tag{20}$$

$$E^{ij}_{\alpha\beta} = -E^{ji}_{\alpha\beta} = -E^{ij}_{\beta\alpha} \tag{21}$$

The lagrangian gauge generators R^i_μ form an open gauge algebra if $E^{ij}_{\alpha\beta} \neq 0$ [8, 5, 9, 6, 10]. If $E^{ij}_{\alpha\beta} = 0$, the gauge algebra is said to be closed[1].

The tensors $T^\gamma_{\alpha\beta}$ and $E^{ij}_{\alpha\beta}$ are the so-called second-order gauge structure functions of the gauge algebra in the lagrangian formalism. The gauge generators R^i_μ are the first-order structure functions of the lagrangian gauge algebra. The zeroth-order structure function is the action S itself [6].

Since ϵ^μ_1 and ϵ^ν_2 are arbitrary functions of time, then from (17) we obtain the following identities in the velocity phase space TQ:

$$\frac{\partial R^i_\mu}{\partial q^j} R^j_\nu - \frac{\partial R^i_\nu}{\partial q^j} R^j_\mu + \left(\frac{\partial R^i_\mu}{\partial \dot{q}^j} \frac{\partial R^j_\nu}{\partial q^k} - \frac{\partial R^i_\nu}{\partial \dot{q}^j} \frac{\partial R^j_\mu}{\partial q^k} \right) \dot{q}^k \equiv T^\gamma_{\mu\nu} R^i_\gamma - E^{ij}_{\mu\nu}\alpha_j \tag{22}$$

$$\frac{\partial R^i_\mu}{\partial \dot{q}^j} \frac{\partial R^j_\nu}{\partial \dot{q}^k} - \frac{\partial R^i_\nu}{\partial \dot{q}^j} \frac{\partial R^j_\mu}{\partial \dot{q}^k} \equiv E^{ij}_{\mu\nu} W_{jk} \tag{23}$$

[1]Notice that by adding trivial gauge transformations to the non-trivial gauge transformations (11) a closed gauge algebra can easily be converted into an "open" gauge algebra. However, in this case the algebra is not really open but "hidden closed". I would like to thank the referee for pointing this out.

$$\frac{\partial R^i_\mu}{\partial \dot{q}^j} R^j_\nu \equiv 0 \tag{24}$$

It is not difficult to prove that the identities (22-24) are equivalent to the so-called second-order relations of the lagrangian gauge algebra given in [6, 10].

From (5,6) and (24) it follows [11] that there exist some functions $b^{ij}_\mu(q, \dot{q})$ such that:

$$\frac{\partial R^i_\mu}{\partial \dot{q}^j} \equiv b^{ik}_\mu W_{kj} \tag{25}$$

Substituting (25) into (23) we find the following identities:

$$(E^{ij}_{\mu\nu} - b^{im}_\mu W_{mn} b^{nj}_\nu + b^{im}_\nu W_{mn} b^{nj}_\mu) W_{jk} \equiv 0 \tag{26}$$

From (6), (9), (22) and (23) one can see that the structure functions $E^{ij}_{\mu\nu}$ are not uniquely determined. In general, one can write:

$$E^{ij}_{\mu\nu} = b^{im}_\mu W_{mn} b^{nj}_\nu - b^{im}_\nu W_{mn} b^{nj}_\mu + e^{\alpha\beta}_{\mu\nu}(R^i_\alpha R^j_\beta - R^i_\beta R^j_\alpha) \tag{27}$$

where $e^{\alpha\beta}_{\mu\nu}(q, \dot{q})$ are arbitrary functions on TQ that are antisymmetric in the lower indexes:

$$e^{\alpha\beta}_{\mu\nu} = -e^{\alpha\beta}_{\nu\mu} \tag{28}$$

From (25) and (27) it immediately follows that:

$$\frac{\partial R^i_\alpha}{\partial \dot{q}^k} E^{kj}_{\beta\gamma} + \frac{\partial R^i_\beta}{\partial \dot{q}^k} E^{kj}_{\gamma\alpha} + \frac{\partial R^i_\gamma}{\partial \dot{q}^k} E^{kj}_{\alpha\beta} + \frac{\partial R^j_\alpha}{\partial \dot{q}^k} E^{ki}_{\beta\gamma} + \frac{\partial R^j_\beta}{\partial \dot{q}^k} E^{ki}_{\gamma\alpha} + \frac{\partial R^j_\gamma}{\partial \dot{q}^k} E^{ki}_{\alpha\beta} \equiv 0 \tag{29}$$

The third-order relations of the lagrangian gauge algebra can be derived from the Jacobi identities [5, 9, 6, 10]:

$$[\delta_1, [\delta_2, \delta_3]] + [\delta_2, [\delta_3, \delta_1]] + [\delta_3, [\delta_1, \delta_2]] \equiv 0 \tag{30}$$

From (30), using (11, 12, 17-24) and (29), we obtain the following identities:

$$R^i_\rho A^\rho_{\alpha\beta\gamma} + B^{ij}_{\alpha\beta\gamma} L_j \equiv 0 \tag{31}$$

$$-R^i_\eta R^k_\alpha \frac{\partial T^\eta_{\beta\gamma}}{\partial \dot{q}^k} - \frac{\partial R^i_\eta}{\partial \dot{q}^k} R^k_\alpha T^\eta_{\beta\gamma} + \left(\frac{\partial R^i_\beta}{\partial \dot{q}^k} E^{kj}_{\gamma\alpha} + \frac{\partial R^i_\gamma}{\partial \dot{q}^k} E^{kj}_{\alpha\beta} + \frac{\partial R^j_\alpha}{\partial \dot{q}^k} E^{ki}_{\beta\gamma} - R^k_\alpha \frac{\partial E^{ij}_{\beta\gamma}}{\partial \dot{q}^k} \right) L_j \equiv 0 \tag{32}$$

where,

$$
\begin{aligned}
A^{\rho}_{\alpha\beta\gamma} \;\; \equiv \;\; & \frac{1}{3}\Big[T^{\rho}_{\alpha\eta}T^{\eta}_{\beta\gamma}+T^{\rho}_{\beta\eta}T^{\eta}_{\gamma\alpha}+T^{\rho}_{\gamma\eta}T^{\eta}_{\alpha\beta} \\
& -R^{j}_{\alpha}\frac{\partial T^{\rho}_{\beta\gamma}}{\partial q^{j}}-R^{j}_{\beta}\frac{\partial T^{\rho}_{\gamma\alpha}}{\partial q^{j}}-R^{j}_{\gamma}\frac{\partial T^{\rho}_{\alpha\beta}}{\partial q^{j}} \\
& -\dot{R}^{j}_{\alpha}\frac{\partial T^{\rho}_{\beta\gamma}}{\partial \dot{q}^{j}}-\dot{R}^{j}_{\beta}\frac{\partial T^{\rho}_{\gamma\alpha}}{\partial \dot{q}^{j}}-\dot{R}^{j}_{\gamma}\frac{\partial T^{\rho}_{\alpha\beta}}{\partial \dot{q}^{j}}\Big]
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
B^{ij}_{\alpha\beta\gamma} \;\; \equiv \;\; & \frac{1}{3}\Big[E^{ij}_{\alpha\eta}T^{\eta}_{\beta\gamma}+E^{ij}_{\beta\eta}T^{\eta}_{\gamma\alpha}+E^{ij}_{\gamma\eta}T^{\eta}_{\alpha\beta} \\
& -R^{k}_{\alpha}\frac{\partial E^{ij}_{\beta\gamma}}{\partial q^{k}}-R^{k}_{\beta}\frac{\partial E^{ij}_{\gamma\alpha}}{\partial q^{k}}-R^{k}_{\gamma}\frac{\partial E^{ij}_{\alpha\beta}}{\partial q^{k}}-\dot{R}^{k}_{\alpha}\frac{\partial E^{ij}_{\beta\gamma}}{\partial \dot{q}^{k}}-\dot{R}^{k}_{\beta}\frac{\partial E^{ij}_{\gamma\alpha}}{\partial \dot{q}^{k}}-\dot{R}^{k}_{\gamma}\frac{\partial E^{ij}_{\alpha\beta}}{\partial \dot{q}^{k}} \\
& +\frac{\partial R^{i}_{\alpha}}{\partial q^{k}}E^{kj}_{\beta\gamma}+\frac{\partial R^{i}_{\beta}}{\partial q^{k}}E^{kj}_{\gamma\alpha}+\frac{\partial R^{i}_{\gamma}}{\partial q^{k}}E^{kj}_{\alpha\beta}-\frac{\partial R^{j}_{\alpha}}{\partial q^{k}}E^{ki}_{\beta\gamma}-\frac{\partial R^{j}_{\beta}}{\partial q^{k}}E^{ki}_{\gamma\alpha}-\frac{\partial R^{j}_{\gamma}}{\partial q^{k}}E^{ki}_{\alpha\beta} \\
& +\frac{\partial R^{i}_{\alpha}}{\partial \dot{q}^{k}}\dot{E}^{kj}_{\beta\gamma}+\frac{\partial R^{i}_{\beta}}{\partial \dot{q}^{k}}\dot{E}^{kj}_{\gamma\alpha}+\frac{\partial R^{i}_{\gamma}}{\partial \dot{q}^{k}}\dot{E}^{kj}_{\alpha\beta}-\frac{\partial R^{j}_{\alpha}}{\partial \dot{q}^{k}}\dot{E}^{ki}_{\beta\gamma}-\frac{\partial R^{j}_{\beta}}{\partial \dot{q}^{k}}\dot{E}^{ki}_{\gamma\alpha}-\frac{\partial R^{j}_{\gamma}}{\partial \dot{q}^{k}}\dot{E}^{ki}_{\alpha\beta} \\
& +\frac{1}{2}\,\frac{d}{dt}\left(\frac{\partial R^{j}_{\alpha}}{\partial \dot{q}^{k}}E^{ki}_{\beta\gamma}+\frac{\partial R^{j}_{\beta}}{\partial \dot{q}^{k}}E^{ki}_{\gamma\alpha}+\frac{\partial R^{j}_{\gamma}}{\partial \dot{q}^{k}}E^{ki}_{\alpha\beta}-\frac{\partial R^{i}_{\alpha}}{\partial \dot{q}^{k}}E^{kj}_{\beta\gamma}-\frac{\partial R^{i}_{\beta}}{\partial \dot{q}^{k}}E^{kj}_{\gamma\alpha}-\frac{\partial R^{i}_{\gamma}}{\partial \dot{q}^{k}}E^{kj}_{\alpha\beta}\right)\Big]
\end{aligned} \tag{34}
$$

Since, by assumption, the gauge generators R^{i}_{ρ} are irreducible, then from (31) it follows that there must exist some functions $D^{i\rho}_{\alpha\beta\gamma}$ such that [6, 10]:

$$
A^{\rho}_{\alpha\beta\gamma} \equiv D^{i\rho}_{\alpha\beta\gamma}L_i \tag{35}
$$

The identities (35) are the so-called third-order relations of the lagrangian gauge algebra [6]. $D^{i\rho}_{\alpha\beta\gamma}$ are the third-order gauge structure functions in the lagrangian formalism.

From (35) we see that the third-order structure functions are not uniquely determined. Indeed, if $D^{i\rho}_{\alpha\beta\gamma}(q,\dot{q})$ satisfy (35), then using (33) and (2) we can rewrite (35) as identities in TQ:

$$
\begin{aligned}
& \frac{1}{3}\Bigg[T^{\rho}_{\alpha\eta}T^{\eta}_{\beta\gamma}+T^{\rho}_{\beta\eta}T^{\eta}_{\gamma\alpha}+T^{\rho}_{\gamma\eta}T^{\eta}_{\alpha\beta}-R^{j}_{\alpha}\frac{\partial T^{\rho}_{\beta\gamma}}{\partial q^{j}}-R^{j}_{\beta}\frac{\partial T^{\rho}_{\gamma\alpha}}{\partial q^{j}}-R^{j}_{\gamma}\frac{\partial T^{\rho}_{\alpha\beta}}{\partial q^{j}} \\
& \qquad -\left(\frac{\partial R^{j}_{\alpha}}{\partial q^{l}}\frac{\partial T^{\rho}_{\beta\gamma}}{\partial \dot{q}^{j}}+\frac{\partial R^{j}_{\beta}}{\partial q^{l}}\frac{\partial T^{\rho}_{\gamma\alpha}}{\partial \dot{q}^{j}}+\frac{\partial R^{j}_{\gamma}}{\partial q^{l}}\frac{\partial T^{\rho}_{\alpha\beta}}{\partial \dot{q}^{j}}\right)\dot{q}^{l}\Bigg] \;\; \equiv \;\; -D^{i\rho}_{\alpha\beta\gamma}\alpha_i
\end{aligned} \tag{36}
$$

$$
-\frac{1}{3}\left[\frac{\partial R^{j}_{\alpha}}{\partial \dot{q}^{k}}\frac{\partial T^{\rho}_{\beta\gamma}}{\partial \dot{q}^{j}}+\frac{\partial R^{j}_{\beta}}{\partial \dot{q}^{k}}\frac{\partial T^{\rho}_{\gamma\alpha}}{\partial \dot{q}^{j}}+\frac{\partial R^{j}_{\gamma}}{\partial \dot{q}^{k}}\frac{\partial T^{\rho}_{\alpha\beta}}{\partial \dot{q}^{j}}\right] \equiv D^{i\rho}_{\alpha\beta\gamma}W_{ik} \tag{37}
$$

From (36, 37) and (6, 9) we conclude that any function $\tilde{D}^{i\rho}_{\alpha\beta\gamma}$:

$$\tilde{D}^{i\rho}_{\alpha\beta\gamma} = D^{i\rho}_{\alpha\beta\gamma} + d^{\rho\delta}_{\alpha\beta\gamma} R^i_\delta \tag{38}$$

must also be a solution of (36, 37). The quantities $d^{\rho\delta}_{\alpha\beta\gamma}$ are arbitrary functions.

Finally, from (31) and (35), using the properties of irreducibility (7) and completeness (13,14), one can obtain the fourth-order relations of the lagrangian gauge algebra in the form [10]:

$$R^i_\rho D^{j\rho}_{\alpha\beta\gamma} - R^j_\rho D^{i\rho}_{\alpha\beta\gamma} + B^{ij}_{\alpha\beta\gamma} \equiv M^{ijk}_{\alpha\beta\gamma} L_k \tag{39}$$

The tensors $M^{ijk}_{\alpha\beta\gamma}$ are the fourth-order gauge structure functions in the lagrangian formalism.

Higher order gauge structure relations can be obtained by commuting a higher number of infinitesimal gauge transformations. All the lagrangian gauge structure relations are encoded in the so-called classical master equation obeyed by the field-antifield action functional [6, 10]. The lagrangian gauge algebra is characterized by the whole set of gauge structure functions.

The existence of the lagrangian gauge structure functions $D^{i\rho}_{\alpha\beta\gamma}$ and $M^{ijk}_{\alpha\beta\gamma}$ has been proven using an axiomatic approach [6, 10]. In this paper we aim to construct these functions explicitly.

With this purpose in mind, let us develop the hamiltonian formulation for the action functional (1). From (5) it follows that our system has m independent primary constraints G_μ ($\mu = 1, 2, ..., m$). Following Dirac's method [12], these primary constraints are obtained from the relations that define the canonical momenta:

$$p_i = \frac{\partial L}{\partial \dot{q}^i}(q, \dot{q}) \tag{40}$$

From (40) and (5) we see that:

$$rank \| \frac{\partial G_\mu}{\partial p_i} \| = m \tag{41}$$

The set of hamiltonian constraints:

$$G_\mu(q, p) = 0 \tag{42}$$

defines a submanifold of the momentum phase space T^*Q which is denoted by $\mathcal{M}$.

The canonical Hamiltonian $H_c(q, p)$ is any function in the momentum phase space T^*Q satisfying the following relation:

$$\mathrm{FL}^* H_c \equiv H_c \left(q, \frac{\partial L}{\partial \dot{q}}(q, \dot{q}) \right) = \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L \tag{43}$$

Following [13] FL is the application fiber derivative of the Lagrangian of the tangent bundle TQ on the cotangent bundle T^*Q,

$$\mathrm{FL}:\ TQ \longrightarrow T^*Q$$

given by $\mathrm{FL}(q,\dot{q}) = (q,p)$ as defined by (40). FL^* is the pullback application.

For the primary hamiltonian constraints G_μ we have:

$$\mathrm{FL}^* G_\mu = G_\mu \left(q, \frac{\partial L}{\partial \dot{q}}(q,\dot{q}) \right) \equiv 0 \tag{44}$$

Differentiating (44) with respect to $\dot{q}^j$ we obtain:

$$\mathrm{FL}^* \frac{\partial G_\mu}{\partial p_i} W_{ij} \equiv 0 \tag{45}$$

Equations (45), (41), (6) and (7) allow us to identify $\mathrm{FL}^* \frac{\partial G_\mu}{\partial p_i}$ with the irreducible gauge generators R^i_μ:

$$R^i_\mu(q,\dot{q}) = \mathrm{FL}^* \frac{\partial G_\mu}{\partial p_i} \tag{46}$$

In other words, one can always choose the constraints G_μ in such a way that the relations (46) are true.

From (46) and (44) it also follows that:

$$-R^j_\mu \frac{\partial^2 L}{\partial q^i \partial \dot{q}^j} = \mathrm{FL}^* \frac{\partial G_\mu}{\partial q^i} \tag{47}$$

From the definition of the canonical Hamiltonian (43) we can also derive the following relations [11]:

$$\mathrm{FL}^* \frac{\partial H_c}{\partial p_i} = \dot{q}^i - \lambda_\mu(q,\dot{q}) R^i_\mu(q,\dot{q}) \tag{48}$$

$$\mathrm{FL}^* \frac{\partial H_c}{\partial q^i} = -\frac{\partial L}{\partial q^i} + \lambda_\mu(q,\dot{q}) R^j_\mu(q,\dot{q}) \frac{\partial^2 L}{\partial \dot{q}^j \partial q^i} \tag{49}$$

Let us consider the Poisson brackets among the primary hamiltonian constraints. From (46) and (47) we obtain:

$$\mathrm{FL}^*\{G_\mu, G_\nu\} \equiv -R^i_\mu B_{ij} R^j_\nu \tag{50}$$

where,

$$B_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} - \frac{\partial^2 L}{\partial \dot{q}^j \partial q^i} \tag{51}$$

For the Poisson brackets between the constraints and the canonical Hamiltonian we have:

$$\mathrm{FL}^*\{H_c, G_\mu\} \equiv -R^i_\mu \alpha_i - R^i_\mu B_{ij} R^j_\nu \lambda_\nu \tag{52}$$

Differentiating (9) with respect to $\dot{q}^j$ and using the identities (6) we obtain the following identities in TQ:

$$\frac{\partial R^i_\alpha}{\partial \dot{q}^j}\alpha_i - R^i_\alpha B_{ij} + \dot{q}^l \frac{\partial R^i_\alpha}{\partial q^l} W_{ij} \equiv 0 \tag{53}$$

Multiplying (53) by R^j_β and using (24) and (6) we obtain the identities:

$$R^i_\alpha B_{ij} R^j_\beta \equiv 0 \tag{54}$$

Therefore, from (50), (52), (9) and (54) we obtain the following identities in the velocity phase space TQ:

$$\mathrm{FL}^*\{G_\mu, G_\nu\} \equiv 0 \tag{55}$$

$$\mathrm{FL}^*\{H_c, G_\mu\} \equiv 0 \tag{56}$$

or equivalently, in the cotangent manifold T^*Q:

$$\{G_\mu, G_\nu\} \equiv C^\alpha_{\mu\nu} G_\alpha \tag{57}$$

$$\{H_c, G_\mu\} \equiv V^\alpha_\mu G_\alpha \tag{58}$$

where $C^\alpha_{\mu\nu}(q,p)$ and $V^\alpha_\mu(q,p)$ are funtions in T^*Q.

From the identities (57) we conclude that all the hamiltonian constraints G_μ ($\mu = 1, 2, ..., m$) are first-class constraints [12]. From (58) we conclude that our system has no secondary constraints [12].

The identities (57) are the first-order relations of the gauge algebra in the hamiltonian formalism [2, 14]. The constraints G_μ are also called the zeroth-order hamiltonian structure functions [2, 14]. The first-order hamiltonian structure functions are the functions $C^\alpha_{\mu\nu}$ in (57) [2, 14].

The second-order relations of the hamiltonian gauge algebra follow from the Jacobi identities:

$$\{\{G_\alpha, G_\beta\}, G_\gamma\} + \{\{G_\beta, G_\gamma\}, G_\alpha\} + \{\{G_\gamma, G_\alpha\}, G_\beta\} \equiv 0 \tag{59}$$

Indeed, from (59) and (57) it follows that:

$$\left(\{C^\eta_{\alpha\beta}, G_\gamma\} + \{C^\eta_{\beta\gamma}, G_\alpha\} + \{C^\eta_{\gamma\alpha}, G_\beta\} - C^\delta_{\alpha\beta}C^\eta_{\gamma\delta} - C^\delta_{\beta\gamma}C^\eta_{\alpha\delta} - C^\delta_{\gamma\alpha}C^\eta_{\beta\delta}\right) G_\eta \equiv 0 \tag{60}$$

From the irreducibility of the constraints (41) it follows [2] that there must exist some functions $J^{\eta\sigma}_{\alpha\beta\gamma}$ in the momentum phase space T^*Q such that:

$$\{C^\eta_{\alpha\beta}, G_\gamma\} + \{C^\eta_{\beta\gamma}, G_\alpha\} + \{C^\eta_{\gamma\alpha}, G_\beta\} - C^\delta_{\alpha\beta}C^\eta_{\gamma\delta} - C^\delta_{\beta\gamma}C^\eta_{\alpha\delta} - C^\delta_{\gamma\alpha}C^\eta_{\beta\delta} \equiv J^{\eta\sigma}_{\alpha\beta\gamma} G_\sigma \tag{61}$$

or equivalently:

$$\mathrm{FL}^*\left(\{C^\eta_{\alpha\beta},G_\gamma\}+\{C^\eta_{\beta\gamma},G_\alpha\}+\{C^\eta_{\gamma\alpha},G_\beta\}-C^\delta_{\alpha\beta}C^\eta_{\gamma\delta}-C^\delta_{\beta\gamma}C^\eta_{\alpha\delta}-C^\delta_{\gamma\alpha}C^\eta_{\beta\delta}\right)\equiv 0 \tag{62}$$

The identities (61) are the so-called second-order relations of the gauge algebra in the hamiltonian formalism [2, 14]. The functions $J^{\eta\sigma}_{\alpha\beta\gamma}$ are the second-order hamiltonian structure functions.

Our purpose in this paper is to write the lagrangian gauge structure functions (R^i_μ, $T^\gamma_{\alpha\beta}$, $E^{ij}_{\alpha\beta}$, $D^{i\rho}_{\alpha\beta\gamma}$ and $M^{ijk}_{\alpha\beta\gamma}$) explicitly in terms of the hamiltonian structure functions G_μ, $C^\gamma_{\alpha\beta}$ and their derivatives. As we shall see, knowledge of the functional dependence of $J^{\eta\sigma}_{\alpha\beta\gamma}$ or other higher order hamiltonian structure functions is not required for this.

Equations (46) give us the lagrangian gauge generators R^i_μ.

From (46), (40) and (3) it immediately follows that:

$$\frac{\partial R^i_\mu}{\partial \dot q^k}\equiv W_{kl}\mathrm{FL}^*\frac{\partial^2 G_\mu}{\partial p_l\partial p_i} \tag{63}$$

$$\frac{\partial R^i_\mu}{\partial q^k}\equiv \mathrm{FL}^*\frac{\partial^2 G_\mu}{\partial q^k\partial p_i}+\frac{\partial^2 L}{\partial q^k\partial \dot q^l}\mathrm{FL}^*\frac{\partial^2 G_\mu}{\partial p_l\partial p_i} \tag{64}$$

Substituting (63, 64) into (22-24) we find:

$$\begin{aligned}\mathrm{FL}^*\left(\frac{\partial^2 G_\mu}{\partial p_i\partial q^j}\frac{\partial G_\nu}{\partial p_j}-\frac{\partial^2 G_\nu}{\partial p_i\partial q^j}\frac{\partial G_\mu}{\partial p_j}\right)+\mathrm{FL}^*\left(\frac{\partial^2 G_\mu}{\partial p_i\partial p_l}\frac{\partial G_\nu}{\partial p_j}-\frac{\partial^2 G_\nu}{\partial p_i\partial p_l}\frac{\partial G_\mu}{\partial p_j}\right)\frac{\partial^2 L}{\partial \dot q^l\partial q^j}+&\\ +\mathrm{FL}^*\left(\frac{\partial^2 G_\mu}{\partial p_i\partial p_l}\frac{\partial^2 G_\nu}{\partial p_m\partial p_j}-\frac{\partial^2 G_\nu}{\partial p_i\partial p_l}\frac{\partial^2 G_\mu}{\partial p_m\partial p_j}\right)W_{lm}\frac{\partial^2 L}{\partial \dot q^j\partial q^k}\dot q^k+&\\ +\mathrm{FL}^*\left(\frac{\partial^2 G_\mu}{\partial p_i\partial p_l}\frac{\partial^2 G_\nu}{\partial p_m\partial q^k}-\frac{\partial^2 G_\nu}{\partial p_i\partial p_l}\frac{\partial^2 G_\mu}{\partial p_m\partial q^k}\right)\dot q^k W_{lm}\equiv T^\gamma_{\mu\nu}R^i_\gamma-E^{ij}_{\mu\nu}\alpha_j&\end{aligned} \tag{65}$$

$$\left(\mathrm{FL}^*\frac{\partial^2 G_\mu}{\partial p_i\partial p_l}W_{lm}\mathrm{FL}^*\frac{\partial^2 G_\nu}{\partial p_m\partial p_j}-\mathrm{FL}^*\frac{\partial^2 G_\nu}{\partial p_i\partial p_l}W_{lm}\mathrm{FL}^*\frac{\partial^2 G_\mu}{\partial p_m\partial p_j}\right)W_{jk}\equiv E^{ij}_{\mu\nu}W_{jk} \tag{66}$$

$$\mathrm{FL}^*\frac{\partial^2 G_\mu}{\partial p_i\partial p_k}W_{kj}R^j_\nu\equiv 0 \tag{67}$$

Substituting (51) and (46, 47) into (65) and using (4) we see that the identities (65) can be rewritten as:

$$\begin{aligned}\mathrm{FL}^*\frac{\partial}{\partial p_i}\{G_\mu,G_\nu\}-\left(\mathrm{FL}^*\frac{\partial^2 G_\mu}{\partial p_i\partial p_l}W_{lm}\mathrm{FL}^*\frac{\partial^2 G_\nu}{\partial p_m\partial p_j}-\mathrm{FL}^*\frac{\partial^2 G_\nu}{\partial p_i\partial p_l}W_{lm}\mathrm{FL}^*\frac{\partial^2 G_\mu}{\partial p_m\partial p_j}\right)\alpha_j&\\ +\mathrm{FL}^*\frac{\partial^2 G_\mu}{\partial p_i\partial_l}\left(B_{lj}\mathrm{FL}^*\frac{\partial G_\nu}{\partial p_j}+W_{lm}\mathrm{FL}^*\frac{\partial^2 G_\nu}{\partial p_m\partial p_j}\frac{\partial L}{\partial q^j}+W_{lm}\mathrm{FL}^*\frac{\partial^2 G_\nu}{\partial p_m\partial q^k}\dot q^k\right)&\end{aligned}$$

$$-\mathrm{FL}^*\frac{\partial^2 G_\nu}{\partial p_i \partial_l}\left(B_{lj}\mathrm{FL}^*\frac{\partial G_\mu}{\partial p_j}+W_{lm}\mathrm{FL}^*\frac{\partial^2 G_\mu}{\partial p_m \partial p_j}\frac{\partial L}{\partial q^j}+W_{lm}\mathrm{FL}^*\frac{\partial^2 G_\mu}{\partial p_m \partial q^k}\dot{q}^k\right)$$
$$\equiv T^\gamma_{\mu\nu}R^i_\gamma - E^{ij}_{\mu\nu}\alpha_j \quad (68)$$

Notice that:

$$\mathrm{FL}^*\frac{\partial}{\partial p_i}\{G_\mu, G_\nu\} \equiv \mathrm{FL}^*\left(\frac{\partial C^\gamma_{\mu\nu}}{\partial p_i}G_\gamma + C^\gamma_{\mu\nu}\frac{\partial G_\gamma}{\partial p_i}\right) \equiv \mathrm{FL}^* C^\gamma_{\mu\nu}\mathrm{FL}^*\frac{\partial G_\gamma}{\partial p_i} \quad (69)$$

On the other hand, from (53), (63) and (64) it follows that:

$$B_{lj}\mathrm{FL}^*\frac{\partial G_\alpha}{\partial p_j}+W_{lm}\mathrm{FL}^*\frac{\partial^2 G_\alpha}{\partial p_m \partial p_j}\frac{\partial L}{\partial q^j}+W_{lm}\mathrm{FL}^*\frac{\partial^2 G_\alpha}{\partial p_m \partial q^k}\dot{q}^k \equiv 0 \quad (70)$$

Finally, substituting (69) and (70) into (68) we obtain the identities:

$$\mathrm{FL}^* C^\gamma_{\mu\nu}R^i_\gamma - \left(\mathrm{FL}^*\frac{\partial^2 G_\mu}{\partial p_i \partial p_l}W_{lm}\mathrm{FL}^*\frac{\partial^2 G_\nu}{\partial p_m \partial p_j} - \mathrm{FL}^*\frac{\partial^2 G_\nu}{\partial p_i \partial p_l}W_{lm}\mathrm{FL}^*\frac{\partial^2 G_\mu}{\partial p_m \partial p_j}\right)\alpha_j$$
$$\equiv T^\gamma_{\mu\nu}R^i_\gamma - E^{ij}_{\mu\nu}\alpha_j \quad (71)$$

From the identities (71) and (66) it follows that we can make the following identification:

$$T^\gamma_{\mu\nu} = \mathrm{FL}^* C^\gamma_{\mu\nu} \quad (72)$$

$$E^{ij}_{\mu\nu} = \mathrm{FL}^*\frac{\partial^2 G_\mu}{\partial p_i \partial p_l}W_{lm}\mathrm{FL}^*\frac{\partial^2 G_\nu}{\partial p_m \partial p_j} - \mathrm{FL}^*\frac{\partial^2 G_\nu}{\partial p_i \partial p_l}W_{lm}\mathrm{FL}^*\frac{\partial^2 G_\mu}{\partial p_m \partial p_j} \quad (73)$$

Equations (72) and (73) give us the lagrangian second-order gauge structure functions in terms of the hamiltonian first-order structure functions and the second derivatives of the hamiltonian constraints with respect to the canonical momenta.

Our task now is to find an expression for the third-order lagrangian gauge structure functions $D^{i\rho}_{\alpha\beta\gamma}$. From (72) it follows that:

$$\frac{\partial T^\rho_{\alpha\beta}}{\partial q^j} = \mathrm{FL}^*\frac{\partial C^\rho_{\alpha\beta}}{\partial q^j} + \frac{\partial^2 L}{\partial q^j \partial \dot{q}^k}\mathrm{FL}^*\frac{\partial C^\rho_{\alpha\beta}}{\partial p_k} \quad (74)$$

$$\frac{\partial T^\rho_{\alpha\beta}}{\partial \dot{q}^j} = W_{jk}\mathrm{FL}^*\frac{\partial C^\rho_{\alpha\beta}}{\partial p_k} \quad (75)$$

Notice also that using (63, 64) and (2), (4) we can write:

$$\dot{R}^j_\alpha = \mathrm{FL}^*\frac{\partial^2 G_\alpha}{\partial p_j \partial q^l}\dot{q}^l + \mathrm{FL}^*\frac{\partial^2 G_\alpha}{\partial p_j \partial p_l}\frac{\partial L}{\partial q^l} + \mathrm{FL}^*\frac{\partial^2 G_\alpha}{\partial p_j \partial p_k}L_k \quad (76)$$

Substituting (72), (74), (75) and (76) into (33) we obtain:

$$A^{\delta}_{\alpha\beta\gamma} = \frac{1}{3}\Bigg[\mathrm{FL}^*\left(C^{\rho}_{\alpha\eta}C^{\eta}_{\beta\gamma} + C^{\rho}_{\beta\eta}C^{\eta}_{\gamma\alpha} + C^{\rho}_{\gamma\eta}C^{\eta}_{\alpha\beta}\right) - \mathrm{FL}^*\left(\frac{\partial G_\alpha}{\partial p_j}\frac{\partial C^{\rho}_{\beta\gamma}}{\partial q^j} + \frac{\partial G_\beta}{\partial p_j}\frac{\partial C^{\rho}_{\gamma\alpha}}{\partial q^j} + \frac{\partial G_\gamma}{\partial p_j}\frac{\partial C^{\rho}_{\alpha\beta}}{\partial q^j}\right)$$
$$-\frac{\partial^2 L}{\partial q^j\partial\dot q^k}\mathrm{FL}^*\left(\frac{\partial G_\alpha}{\partial p_j}\frac{\partial C^{\rho}_{\beta\gamma}}{\partial p_k} + \frac{\partial G_\beta}{\partial p_j}\frac{\partial C^{\rho}_{\gamma\alpha}}{\partial p_k} + \frac{\partial G_\gamma}{\partial p_j}\frac{\partial C^{\rho}_{\alpha\beta}}{\partial p_k}\right)$$
$$-\dot q^l\left(\mathrm{FL}^*\frac{\partial^2 G_\alpha}{\partial q^l\partial p_j}W_{jk}\mathrm{FL}^*\frac{\partial C^{\rho}_{\beta\gamma}}{\partial p_k} + \mathrm{FL}^*\frac{\partial^2 G_\beta}{\partial q^l\partial p_j}W_{jk}\mathrm{FL}^*\frac{\partial C^{\rho}_{\gamma\alpha}}{\partial p_k} + \mathrm{FL}^*\frac{\partial^2 G_\gamma}{\partial q^l\partial p_j}W_{jk}\mathrm{FL}^*\frac{\partial C^{\rho}_{\alpha\beta}}{\partial p_k}\right)$$
$$-\frac{\partial L}{\partial q^l}\left(\mathrm{FL}^*\frac{\partial^2 G_\alpha}{\partial p_l\partial p_j}W_{jk}\mathrm{FL}^*\frac{\partial C^{\rho}_{\beta\gamma}}{\partial p_k} + \mathrm{FL}^*\frac{\partial^2 G_\beta}{\partial p_l\partial p_j}W_{jk}\mathrm{FL}^*\frac{\partial C^{\rho}_{\gamma\alpha}}{\partial p_k} + \mathrm{FL}^*\frac{\partial^2 G_\gamma}{\partial p_l\partial p_j}W_{jk}\mathrm{FL}^*\frac{\partial C^{\rho}_{\alpha\beta}}{\partial p_k}\right)$$
$$-L_j\left(\mathrm{FL}^*\frac{\partial^2 G_\alpha}{\partial p_j\partial p_l}W_{lk}\mathrm{FL}^*\frac{\partial C^{\rho}_{\beta\gamma}}{\partial p_k} + \mathrm{FL}^*\frac{\partial^2 G_\beta}{\partial p_j\partial p_l}W_{lk}\mathrm{FL}^*\frac{\partial C^{\rho}_{\gamma\alpha}}{\partial p_k} + \mathrm{FL}^*\frac{\partial^2 G_\gamma}{\partial p_j\partial p_l}W_{lk}\mathrm{FL}^*\frac{\partial C^{\rho}_{\alpha\beta}}{\partial p_k}\right)\Bigg] \tag{77}$$

Using (51) and (46,47) we can rewrite (77) as follows:

$$A^{\delta}_{\alpha\beta\gamma} = \frac{1}{3}\Big[\mathrm{FL}^*\left(-\{C^{\rho}_{\alpha\beta}, G_\gamma\} - \{C^{\rho}_{\gamma\alpha}, G_\beta\} - \{C^{\rho}_{\beta\gamma}, G_\alpha\} + C^{\rho}_{\alpha\eta}C^{\eta}_{\beta\gamma} + C^{\rho}_{\beta\eta}C^{\eta}_{\gamma\alpha} + C^{\rho}_{\gamma\eta}C^{\eta}_{\alpha\beta}\right)$$
$$-\mathrm{FL}^*\frac{\partial C^{\rho}_{\alpha\beta}}{\partial p_k}\left(B_{kj}\mathrm{FL}^*\frac{\partial G_\gamma}{\partial p_j} + W_{jk}\mathrm{FL}^*\frac{\partial^2 G_\gamma}{\partial p_l\partial p_j}\frac{\partial L}{\partial q^l} + W_{jk}\mathrm{FL}^*\frac{\partial^2 G_\gamma}{\partial p_j\partial q^l}\dot q^l\right)$$
$$-\mathrm{FL}^*\frac{\partial C^{\rho}_{\gamma\alpha}}{\partial p_k}\left(B_{kj}\mathrm{FL}^*\frac{\partial G_\beta}{\partial p_j} + W_{jk}\mathrm{FL}^*\frac{\partial^2 G_\beta}{\partial p_l\partial p_j}\frac{\partial L}{\partial q^l} + W_{jk}\mathrm{FL}^*\frac{\partial^2 G_\beta}{\partial p_j\partial q^l}\dot q^l\right)$$
$$-\mathrm{FL}^*\frac{\partial C^{\rho}_{\beta\gamma}}{\partial p_k}\left(B_{kj}\mathrm{FL}^*\frac{\partial G_\alpha}{\partial p_j} + W_{jk}\mathrm{FL}^*\frac{\partial^2 G_\alpha}{\partial p_l\partial p_j}\frac{\partial L}{\partial q^l} + W_{jk}\mathrm{FL}^*\frac{\partial^2 G_\alpha}{\partial p_j\partial q^l}\dot q^l\right)$$
$$-L_j\left(\mathrm{FL}^*\frac{\partial^2 G_\alpha}{\partial p_j\partial p_l}W_{lk}\mathrm{FL}^*\frac{\partial C^{\rho}_{\beta\gamma}}{\partial p_k} + \mathrm{FL}^*\frac{\partial^2 G_\beta}{\partial p_j\partial p_l}W_{lk}\mathrm{FL}^*\frac{\partial C^{\rho}_{\gamma\alpha}}{\partial p_k} + \mathrm{FL}^*\frac{\partial^2 G_\gamma}{\partial p_j\partial p_l}W_{lk}\mathrm{FL}^*\frac{\partial C^{\rho}_{\alpha\beta}}{\partial p_k}\right)\Big] \tag{78}$$

Finally, using (62) and (70) we obtain:

$$A^{\delta}_{\alpha\beta\gamma} = -\frac{1}{3}L_j\left(\mathrm{FL}^*\frac{\partial^2 G_\alpha}{\partial p_j\partial p_l}W_{lk}\mathrm{FL}^*\frac{\partial C^{\rho}_{\beta\gamma}}{\partial p_k} + \mathrm{FL}^*\frac{\partial^2 G_\beta}{\partial p_j\partial p_l}W_{lk}\mathrm{FL}^*\frac{\partial C^{\rho}_{\gamma\alpha}}{\partial p_k} + \mathrm{FL}^*\frac{\partial^2 G_\gamma}{\partial p_j\partial p_l}W_{lk}\mathrm{FL}^*\frac{\partial C^{\rho}_{\alpha\beta}}{\partial p_k}\right) \tag{79}$$

From (35) and (79) we see that the third-order lagrangian gauge structure functions $D^{i\rho}_{\alpha\beta\gamma}$ can be written as:

$$D^{i\rho}_{\alpha\beta\gamma} = -\frac{1}{3}\left(\mathrm{FL}^*\frac{\partial^2 G_\alpha}{\partial p_i\partial p_j}W_{jk}\mathrm{FL}^*\frac{\partial C^{\rho}_{\beta\gamma}}{\partial p_k} + \mathrm{FL}^*\frac{\partial^2 G_\beta}{\partial p_i\partial p_j}W_{jk}\mathrm{FL}^*\frac{\partial C^{\rho}_{\gamma\alpha}}{\partial p_k} + \mathrm{FL}^*\frac{\partial^2 G_\gamma}{\partial p_i\partial p_j}W_{jk}\mathrm{FL}^*\frac{\partial C^{\rho}_{\alpha\beta}}{\partial p_k}\right) \tag{80}$$

The derivation presented above can also be viewed as a proof by construction of the existence of the gauge structure functions $D^{i\rho}_{\alpha\beta\gamma}$.

Using (6), (63), (73) and (75) one can easily prove the identities (32).

Let us now find the fourth-order lagrangian gauge structure functions $M^{ijk}_{\alpha\beta\gamma}$. For that, we need to write the left-hand-side of (39) in terms of hamiltonian quantities. After some lengthy calculations we find:

$$
\begin{aligned}
R^i_\rho D^{j\rho}_{\alpha\beta\gamma} - R^j_\rho D^{i\rho}_{\alpha\beta\gamma} \equiv & \\
& -\frac{1}{3}\Big[E^{ij}_{\alpha\eta}T^\eta_{\beta\gamma} + E^{ij}_{\beta\eta}T^\eta_{\gamma\alpha} + E^{ij}_{\gamma\eta}T^\eta_{\alpha\beta} \\
& +\mathrm{FL}^*\left(\frac{\partial^2}{\partial p_i\partial p_k}\{G_\alpha,G_\beta\}\frac{\partial^2 G_\gamma}{\partial p_l\partial p_j} - \frac{\partial^2}{\partial p_j\partial p_k}\{G_\alpha,G_\beta\}\frac{\partial^2 G_\gamma}{\partial p_l\partial p_i}\right)W_{kl} \\
& +\mathrm{FL}^*\left(\frac{\partial^2}{\partial p_i\partial p_k}\{G_\beta,G_\gamma\}\frac{\partial^2 G_\alpha}{\partial p_l\partial p_j} - \frac{\partial^2}{\partial p_j\partial p_k}\{G_\beta,G_\gamma\}\frac{\partial^2 G_\alpha}{\partial p_l\partial p_i}\right)W_{kl} \\
& +\mathrm{FL}^*\left(\frac{\partial^2}{\partial p_i\partial p_k}\{G_\gamma,G_\alpha\}\frac{\partial^2 G_\beta}{\partial p_l\partial p_j} - \frac{\partial^2}{\partial p_j\partial p_k}\{G_\gamma,G_\alpha\}\frac{\partial^2 G_\beta}{\partial p_l\partial p_j}\right)W_{kl}\Big]
\end{aligned}
\tag{81}
$$

$$
\begin{aligned}
B^{ij}_{\alpha\beta\gamma} \equiv\ & \frac{1}{3}\Big[E^{ij}_{\alpha\eta}T^\eta_{\beta\gamma} + E^{ij}_{\beta\eta}T^\eta_{\gamma\alpha} + E^{ij}_{\gamma\eta}T^\eta_{\alpha\beta} \\
& +\mathrm{FL}^*\left(\frac{\partial^2}{\partial p_i\partial p_k}\{G_\alpha,G_\beta\}\frac{\partial^2 G_\gamma}{\partial p_l\partial p_j} - \frac{\partial^2}{\partial p_j\partial p_k}\{G_\alpha,G_\beta\}\frac{\partial^2 G_\gamma}{\partial p_l\partial p_i}\right)W_{kl} \\
& +\mathrm{FL}^*\left(\frac{\partial^2}{\partial p_i\partial p_k}\{G_\beta,G_\gamma\}\frac{\partial^2 G_\alpha}{\partial p_l\partial p_j} - \frac{\partial^2}{\partial p_j\partial p_k}\{G_\beta,G_\gamma\}\frac{\partial^2 G_\alpha}{\partial p_l\partial p_i}\right)W_{kl} \\
& +\mathrm{FL}^*\left(\frac{\partial^2}{\partial p_i\partial p_k}\{G_\gamma,G_\alpha\}\frac{\partial^2 G_\beta}{\partial p_l\partial p_j} - \frac{\partial^2}{\partial p_j\partial p_k}\{G_\gamma,G_\alpha\}\frac{\partial^2 G_\beta}{\partial p_l\partial p_j}\right)W_{kl}\Big] \\
& -\frac{1}{3}L_k\Bigg[\mathrm{FL}^*\frac{\partial^2 G_\alpha}{\partial p_k\partial p_l}\frac{\partial E^{ij}_{\beta\gamma}}{\partial \dot q^l} + \mathrm{FL}^*\frac{\partial^2 G_\beta}{\partial p_k\partial p_l}\frac{\partial E^{ij}_{\gamma\alpha}}{\partial \dot q^l} + \mathrm{FL}^*\frac{\partial^2 G_\gamma}{\partial p_k\partial p_l}\frac{\partial E^{ij}_{\alpha\beta}}{\partial \dot q^l} \\
& +\mathrm{FL}^*\frac{\partial^3 G_\alpha}{\partial p_k\partial p_m\partial p_n}\left(\frac{\partial R^i_\beta}{\partial \dot q^m}\frac{\partial R^j_\gamma}{\partial \dot q^n} - \frac{\partial R^i_\gamma}{\partial \dot q^m}\frac{\partial R^j_\beta}{\partial \dot q^n}\right) \\
& +\mathrm{FL}^*\frac{\partial^3 G_\beta}{\partial p_k\partial p_m\partial p_n}\left(\frac{\partial R^i_\gamma}{\partial \dot q^m}\frac{\partial R^j_\alpha}{\partial \dot q^n} - \frac{\partial R^i_\alpha}{\partial \dot q^m}\frac{\partial R^j_\gamma}{\partial \dot q^n}\right) \\
& +\mathrm{FL}^*\frac{\partial^3 G_\gamma}{\partial p_k\partial p_m\partial p_n}\left(\frac{\partial R^i_\alpha}{\partial \dot q^m}\frac{\partial R^j_\beta}{\partial \dot q^n} - \frac{\partial R^i_\beta}{\partial \dot q^m}\frac{\partial R^j_\alpha}{\partial \dot q^n}\right)\Bigg]
\end{aligned}
\tag{82}
$$

From (39) and (81, 82) we see that it is possible to make the following identification:

$$
\begin{aligned}
M^{ijk}_{\alpha\beta\gamma} \equiv & -\frac{1}{3}\left[\mathrm{FL}^*\frac{\partial^2 G_\alpha}{\partial p_k \partial p_l}\frac{\partial E^{ij}_{\beta\gamma}}{\partial \dot{q}^l} + \mathrm{FL}^*\frac{\partial^2 G_\beta}{\partial p_k \partial p_l}\frac{\partial E^{ij}_{\gamma\alpha}}{\partial \dot{q}^l} + \mathrm{FL}^*\frac{\partial^2 G_\gamma}{\partial p_k \partial p_l}\frac{\partial E^{ij}_{\alpha\beta}}{\partial \dot{q}^l}\right. \\
& +\mathrm{FL}^*\frac{\partial^3 G_\alpha}{\partial p_k \partial p_m \partial p_n}\left(\frac{\partial R^i_\beta}{\partial \dot{q}^m}\frac{\partial R^j_\gamma}{\partial \dot{q}^n} - \frac{\partial R^i_\gamma}{\partial \dot{q}^m}\frac{\partial R^j_\beta}{\partial \dot{q}^n}\right) \\
& +\mathrm{FL}^*\frac{\partial^3 G_\beta}{\partial p_k \partial p_m \partial p_n}\left(\frac{\partial R^i_\gamma}{\partial \dot{q}^m}\frac{\partial R^j_\alpha}{\partial \dot{q}^n} - \frac{\partial R^i_\alpha}{\partial \dot{q}^m}\frac{\partial R^j_\gamma}{\partial \dot{q}^n}\right) \\
& \left.+\mathrm{FL}^*\frac{\partial^3 G_\gamma}{\partial p_k \partial p_m \partial p_n}\left(\frac{\partial R^i_\alpha}{\partial \dot{q}^m}\frac{\partial R^j_\beta}{\partial \dot{q}^n} - \frac{\partial R^i_\beta}{\partial \dot{q}^m}\frac{\partial R^j_\alpha}{\partial \dot{q}^n}\right)\right] \qquad (83)
\end{aligned}
$$

Using (63) and (73) we can rewrite (83) in terms of the hamiltonian constraints as follows:

$$
\begin{aligned}
M^{ijk}_{\alpha\beta\gamma} \equiv & -\frac{1}{3}\left[\mathrm{FL}^*\frac{\partial^2 G_\alpha}{\partial p_k \partial p_l}P^{ij}_{l\beta\gamma} + \mathrm{FL}^*\frac{\partial^2 G_\beta}{\partial p_k \partial p_l}P^{ij}_{l\gamma\alpha} + \mathrm{FL}^*\frac{\partial^2 G_\gamma}{\partial p_k \partial p_l}P^{ij}_{l\alpha\beta}\right. \\
& +\mathrm{FL}^*\frac{\partial^3 G_\alpha}{\partial p_k \partial p_m \partial p_n}\left(P^i_{m\beta}P^j_{n\gamma} - P^i_{m\gamma}P^j_{n\beta}\right) \\
& +\mathrm{FL}^*\frac{\partial^3 G_\beta}{\partial p_k \partial p_m \partial p_n}\left(P^i_{m\gamma}P^j_{n\alpha} - P^i_{m\alpha}P^j_{n\gamma}\right) \\
& \left.+\mathrm{FL}^*\frac{\partial^3 G_\gamma}{\partial p_k \partial p_m \partial p_n}\left(P^i_{m\alpha}P^j_{n\beta} - P^i_{m\beta}P^j_{n\alpha}\right)\right] \qquad (84)
\end{aligned}
$$

where,

$$
P^i_{j\mu} = W_{jk}\mathrm{FL}^*\frac{\partial^2 G_\mu}{\partial p_k \partial p_i} \qquad (85)
$$

$$
\begin{aligned}
P^{ij}_{k\mu\nu} = & \frac{\partial W_{lm}}{\partial \dot{q}^k}\mathrm{FL}^*\left(\frac{\partial^2 G_\mu}{\partial p_i \partial p_l}\frac{\partial^2 G_\nu}{\partial p_m \partial p_j} - \frac{\partial^2 G_\nu}{\partial p_i \partial p_l}\frac{\partial^2 G_\mu}{\partial p_m \partial p_j}\right) \\
& +W_{lm}W_{kn}\mathrm{FL}^*\frac{\partial}{\partial p_n}\left(\frac{\partial^2 G_\mu}{\partial p_i \partial p_l}\frac{\partial^2 G_\nu}{\partial p_m \partial p_j} - \frac{\partial^2 G_\nu}{\partial p_i \partial p_l}\frac{\partial^2 G_\mu}{\partial p_m \partial p_j}\right) \qquad (86)
\end{aligned}
$$

The main results presented in this paper are the explicit expressions for the lagrangian gauge structure functions $E^{ij}_{\mu\nu}$, $D^{i\rho}_{\alpha\beta\gamma}$ and $M^{ijk}_{\alpha\beta\gamma}$ given by the equations (73), (80) and (84). These equations show how the higher-order lagrangian structure tensors are determined by the hamiltonian constraints and the hamiltonian first-order structure functions. To determine these lagrangian structure tensors no knowledge of the higher-order hamiltonian structure functions is required. Notice that for the lagrangian gauge algebra to be open

($E^{ij}_{\mu\nu} \neq 0$) it is necessary to have at least two constraints that depend nonlinearly on the momenta. In order to have nonvanishing third-order structure functions $D^{i\rho}_{\alpha\beta\gamma}$, the first-order structure functions $C^{\eta}_{\mu\nu}$ must depend on the canonical momenta p_i. The third derivatives of the constraints with respect to the canonical momenta determine the fourth-order structure tensors $M^{ijk}_{\alpha\beta\gamma}$. The method presented here can be used to obtain tensors of even higher orders.

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